

The Riemann hypothesis - an elementary analytic approach based on complex Laplace transform

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Abstract. An elementary analytic proof of the famous Riemann hypothesis is given. The main "accent" of the proof is a both using of the 2-dimensional double real and complex Laplace integral representations of the Green function $|z|^{-2}$.

1 Introduction.

Let \mathbb{R} and \mathbb{C} be the real and complex number fields, respectively. In [ML] we gave a proof of the **Riemann hypothesis** (Rh for short) based on the following two-dimensional and two-sided **Laplace representation** of the complex **Green function** $G_2(z) = |z|^{-2}$:

$$|z|^{-2} = \int_{\mathbb{C}} e^{z \cdot c} dH_2^*(c), \quad z \in \mathbb{C}^* := \mathbb{C} - \{0\}, \quad (1.1)$$

where $z \cdot c = \operatorname{re}(z)\operatorname{re}(c) + \operatorname{im}(z)\operatorname{im}(c)$ is the usual Euclidean scalar product in $\mathbb{C} = \mathbb{R}^2$, H_2^* is the so called **Hodge measure** (see [ML] for the motivation of the name) and all in the sequel $\operatorname{re}(z)$, $\operatorname{im}(z)$ denotes the real and imaginary part of a complex number z .

To obtain the representation (1.1) we used in [ML] the machine of stochastic analysis based on the Wiener measure w_∞ .

In [MR] we gave another two constructions of H_2^* based on l-adic Wiener measure w_l and l-adic Gibbs measure G_l (see [MR] - for details).

In this paper, basing strongly on some **Pitkannen's idea** (trial) of a quantum mechanics (conformal) proof of Rh (see [Pi]), we will be able - in a trivial way - establish the **elementary complex Laplace representation** $\operatorname{Rep}(\mathbb{C})$ of G_2 : let $\chi_{\mathbb{R}_+}(x)$ be the characteristic function of $\mathbb{R}_+ := [0, +\infty)$ in \mathbb{R} . It is so called **Heaviside function** - important in electricity (see e.g. [LZ, IV.3, Example 1]). Using its, we have the following **trivial formula**

$$\frac{1}{z} = \int_{\mathbb{R}} e^{-zl} \chi_{\mathbb{R}_+}(l) dl = \int_0^\infty e^{-zl} dl = \mathcal{L}_{\mathbb{C}}(\chi_{\mathbb{R}_+})(z) \quad (1.2)$$

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if $\operatorname{re}(z) > 0$.

In fact the formula (1.2) is a little bit formal. The strict proof of (1.2) requires the calculations of the real Laplace transforms of sine and cosine. Thus we have : let $z = u+iv$. Then (see e.g. [LZ])

$$\begin{aligned} \int_0^\infty e^{-zl} dl &= \int_0^\infty e^{-ul} \cos vl dl - i \int_0^\infty e^{-ul} \sin vl dl = \\ &= \mathcal{L}(\cos vl)(u) - i \mathcal{L}(\sin vl)(u) = \frac{u - iv}{u^2 + v^2} = \frac{\tilde{z}}{|z|^2} = \frac{1}{z}. \end{aligned}$$

According to the **Pitkanen's patent** for $|z|^2$ and **Fubini theorem** we get

$$\begin{aligned} \frac{1}{|z|^2} &= \frac{1}{z\tilde{z}} = \int_0^\infty e^{-zl_1} dl_1 \int_0^\infty e^{-\tilde{z}l_2} dl_2 = \\ &= \int \int_{\mathbb{R}_+^2} e^{-[zl_1 + \tilde{z}l_2]} dl_1 dl_2 = \int_{\mathbb{R}_+^2} e^{-\langle z, l \rangle} d^2l, \end{aligned} \tag{1.3}$$

for each $z \in \mathbb{C}$ with $\operatorname{re}(z) > 0$, where $c(x + iy) = x + iy := x - iy$ is the **complex conjugation**. (Let us explain what we mean here by the Pitkanen's patent : in [Pi], he used the methods of the quantum conformal field theory. He tried to realize the famous **Hilbert-Polya conjecture** that complex zeroes ρ_n of ζ should be **eigenvalues** of $1/2 + iH$, where H is a hypothetical **hermitian operator** on a hypothetical Hilbert space \mathcal{H} , i.e. $\rho_n = ev_n(iH + 1/2)$ (with a discrete spectrum), having the decomposition : $H = DD^+$ with such non-hermitian operators D and D^+ with the property that they have zeroes ρ_n and their conjugates as their complex eigenvalues. Thus, we can write : $D(\psi_n) = \rho_n \psi_n$ and $D^+(\psi_n) = \tilde{\rho}_n \psi_n$, for some eigenvectors $\psi_n \in \mathcal{H}$).

Through this paper, we are using the bi-linear form $\langle ., . \rangle : \mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by (1.3), i.e. we put

$$\langle z, l \rangle := zl_1 + c(z)l_2, \tag{1.4}$$

if $z \in \mathbb{C}$ and $l = (l_1, l_2) \in \mathbb{R}^2$.

The existence of the complex Laplace integral representation (1.3) (or the double Fresnel integral (oscillatory) representation):

$$|z|^{-2} = \int \int_{\mathbb{R}_+^2} e^{-\operatorname{re}(z)(l_1+l_2)} \cos(\operatorname{im}(z)(l_1 - l_2)) dl_1 dl_2$$

is a very happy circumstance (event) - it solves the so called - **Zabczyk's problem (question)** (Z.q. for short) : prove - in an elementary way - that the function of the complex variable $|z|^{-2}$ is LHpd (or **positively definite** in the **Toeplitz sense**(see [A])) on \mathbb{R}_+^2 - posed at a Seminar on Stochastic Analysis at IMPAN in Warsaw - during the presentation by the author the so called Hodgkin proof of the Riemann hypothesis. (Let us mention that the same problem was erected by **M. Bożejko** - by a private communication - many years earlier. Thus, it is may be better to say on a **Bożejko-Zabczyk question**(BZq - for short)).

Finally, we remark that similarly like in [MW] we proved (Rh) using the methods of functional analysis and probability theory, in [MH] - the methods of harmonic analysis and arithmetics of (Rh) - the methods of that paper are concentrated (belong) to the Laplace transform theory (important - for example in probabilistic and theory of partial differential equations) and the classical elementary analysis. Thus it is an elementary analysis of the Riemann hypothesis.

2 The Fresnel integrals, Bernstein-Widder theorem and existence of the Hodge measure H_2 .

In [AM] we introduced and considered some particularly useful and simple property of Fourier integrals on \mathbb{R} : the **non-vanishing** of the so called **Fresnel integrals** (cf.e.g. [S, IV.9., Example], [EFI, p. 635], [PEGK]). For a very detailed and deep study of Fresnel integrals see [AC]

Fresnel was probably the first, who considered the following (fundamental in optics) **Fresnel integrals** :

$$F_2(\nu) := \int_0^\infty \sin \nu x^2 dx = \int_0^\infty \cos \nu x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2\nu}},$$

according to some problems concerning the light scattering theory.

In [S], **L. Schwartz** considered the integrals :

$$F(r, \nu) := \int_0^\infty \frac{\sin \nu x}{x^r} dx \text{ for } r > 0 \text{ and } \nu > 0.$$

In particular, the value $F(1, \nu) = \pi/2$ does not depend on ν and

$$F(1/2, \nu) = \int_0^\infty \frac{\cos \nu x}{\sqrt{x}} dx = 2 \int_0^\infty \cos \nu x^2 = 2F_2(\nu) = \sqrt{\frac{\pi}{2\nu}}.$$

Therefore, in the sequel, it would be very convenient to introduce the following integrals : each **Positive Continuous Integrable and (strictly) Decreasing** function $A : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ - we call in the sequel a **PCID-amplitude** (see [AM]) and any $\nu \in \mathbb{R}_+$ - the **frequency**. Then we can consider the integrals :

$$F_s(A, \nu) := \int_0^\infty A(x) \sin \nu x dx \text{ and } F_c(A, \nu) := \int_0^\infty A(x) \cos \nu x dx. \quad (2.5)$$

According to the mentional above fundamental examples, it will be very convenient to call such integrals - the **Fresnel integrals** (associated with an amplitude $A(x)$ and a frequency ν).

(**M. Pluta** has communicated to the author that - in fact in optics - important are rather Fresnel integrals being the functions of the upper limit in the definite integral - by

private communication. But for the purposes of this paper we need only Fresnel integrals as numbers).

Obviously, $F_s(A, \nu)$ and $F_c(A, \nu)$ are thus nothing that the **real** and **imaginary parts** of the **Fourier (ossilatory) integrals** (see e.g. [Ar]), i.e.

$$\mathcal{F}(A, \nu) := \int_{\mathbb{R}} A(x) e^{i\nu x} dx,$$

and , obviously $F_c(A, \nu) = \text{re}(\mathcal{F}(A, \nu))$ and $F_s(A, \nu) = \text{im}(\mathcal{F}(A, \nu))$.

We have the following simple (but extremaly useful) analytic elementary lemma :

Lemma 1 (The Fresnel Lemma.)

The Fresnel integrals of PCID-amplitudes are strongly positive, i.e. for each $\nu > 0$ and each PCID-amplitude $A(x)$ holds:

$$F_s(A, \nu) > 0 \text{ and } F_c(A, \nu) = -\frac{1}{\nu} F_s(A', \nu). \quad (2.6)$$

Proof. We have

$$\begin{aligned} F_s(A, \nu) &= \int_0^\infty A(x) \sin(2\pi\nu x) dx = \sum_{n=0}^\infty \int_{n/\nu}^{(n+1)/\nu} A(x) \sin(2\pi\nu x) dx = \\ &= \left(\sum_{n=0}^\infty (-1)^n A(x_n) \right) P(\nu), \end{aligned}$$

where $P(\nu) = \int_0^{1/\nu} \sin 2\pi\nu x dx > 0$ and a sequence $\{x_n\}$ with $x_n \in [n/2\nu, (n+1)/\nu]$ is determined according to the elementary mean value theorem (see [AM]).

Moreover, integrating by parts we have

$$\begin{aligned} F_c(A, \nu) &= \int_0^\infty A(x) \cos \nu x dx = (A(x) \sin \nu x) / \nu \Big|_0^\infty - \frac{1}{\nu} \int_0^\infty A'(x) \sin \nu x dx = \\ &= -\frac{1}{\nu} F_s(A', \nu). \end{aligned}$$

Since

$$|z|^{-2} = \text{re} \left(\int \int_{\mathbb{R}_+^2} e^{-\langle z, l \rangle} d^2 l \right) = \int \int_{\mathbb{R}_+^2} e^{-x(l_1+l_2)} \cos(y(l_2-l_1)) dl_1 dl_2 =: F_{22}(z), \quad z = x + iy, \quad (2.7)$$

thus $|z|^{-2}$ is nothing that the **double Fresnel integral**. Doing the change of variables in $F_{22}(z)$ (the Jacobi theorem), according to the substitution : $l_1 = u, l_2 - l_1 = v$, we obtain

$$\begin{aligned} F_{22}(z) &= \int \int_{\mathbb{R}_+^2} e^{-x(v+2u)} \cos(yv) du dv = \int_0^\infty e^{-2xu} du \int_0^\infty e^{-xv} \cos(yv) dv = \\ &= F_c(\exp^{-2\text{re}(z)})(0) F_c(\exp^{-\text{re}(z)})(\text{im}(z)) > 0, \end{aligned}$$

according to the **Fubini theorem**.

Let us recall that we have for the disposition the following two **bilinear forms** : the **complex form**

$$\langle z, l \rangle : \mathbb{C}_{++} \times \mathbb{R}_+^2 \longrightarrow \mathbb{C},$$

and the **real form** (the Hilbert (Euclidean) scalar product) $z \cdot l : \mathbb{C} \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$

$$z \cdot l := \operatorname{re}(z)l_1 + \operatorname{im}(z)l_2 \quad , \quad l = (l_1, l_2).$$

For any positive σ -additive **measure** μ on $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$, where $\mathcal{B}(\mathbb{R}_+^2)$ is the σ -field of all **Borel subsets** of \mathbb{R}_+^2 , we can consider two **double Laplace transforms** :

1. the **complex Laplace transform** of a measure μ ,

$$\mathcal{L}_2^+(\mathbb{C})(\mu)(z) := \int \int_{\mathbb{R}_+^2} e^{-\langle z, l \rangle} d\mu(l), \quad (2.8)$$

and

2. the **real Laplace transform** of a measure μ ,

$$\mathcal{L}_2^+(\mathbb{R})(\mu)(z) := \int \int_{\mathbb{R}_+^2} e^{-z \cdot l} d\mu(l) = \hat{\mu}(z). \quad (2.9)$$

Remark 1 *Let us recall (see [ML]) that we have in fact two (completely different) harmonic analysis notions of the **positive definity** (p.d. for short).*

*We say that a function $l : (S, +) \longrightarrow \mathbb{R}_+$, defined on an **abelian semigroup** $(S, +)$ is **positive definite in Laplace-Hankel sense** (LHpD in short - or simply we say on the **semigroup positive-definity**) iff for each real n -tuple $(r_1, \dots, r_n) \in \mathbb{R}^n$ and each semigroup n -tuple $(s_1, \dots, s_n) \in S^n$ holds*

$$(LHpD) \quad \sum_{i=1}^n \sum_{j=1}^n r_i r_j l(s_i + s_j) \geq 0,$$

*whereas, a function $f : (G, +) \longrightarrow \mathbb{C}$, defined on an **abelian group** $(G, +)$ is called the **positive definite in the Fourier-Hermite sense** (FHpd for short - or simply we say on the **group positive-definity**) iff for any n -tuples $g = (g_1, \dots, g_n)$ of elements of G and $c = (c_1, \dots, c_n)$ of complex numbers holds*

$$(FHpd) \quad \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j f(g_i - g_j) \geq 0.$$

The real Laplace transforms $\mathcal{L}_2^+(\mathbb{R})(\mu) = \hat{\mu}$ are obviously LHpD (see also [ML] and [A]). But it is the very surprising fact that the **algebraically-ordered condition** LHpD gives the **characterization** of Laplace transforms on LCA-semigroups (see [ML] and [A, Th. 5.5.4 and p.228]), according to the following deep

Theorem 1 (The Bernstein-Widder theorem for \mathbb{R}^n).

In order that the representation

$$l(x) = \int_{\mathbb{R}^n} e^{x \cdot u} dh_l(u), \quad (2.10)$$

*be valid, where $h_l(u)$ is an **unique non-decreasing function on each factor variable**, it is necessary and sufficient that $l(x)$ is **continuous LHp**d on \mathbb{R}^n .*

Remark 2 *Let us mention the following beautiful **Sierpiński theorem** - at this moment - if $f : (a, b) \rightarrow \mathbb{R}$ is such a function that $-\infty < f(x) \leq \infty (a < x < b)$, is convex and measurable, then $f(x)$ is continuous.*

Theorem 2 (On the existence of the Hodge measure H_2).

For each $z \in \mathbb{C}$ with $\operatorname{re}(z) > 0$ and $\operatorname{im}(z) > 0$, we have

$$\operatorname{Rep}(\mathbb{R}) \quad |z|^{-2} = \int \int_{\mathbb{R}_+^2} e^{-z \cdot l} dH_2(l) = \mathcal{L}_2^+(\mathbb{R})(H_2)(z). \quad (2.11)$$

Proof. We show that the function $g_2(z) = |z|^{-2}$ (the second Green function) is LHp on \mathbb{R}_+^2 , i.e. for each $z = (z_1, \dots, z_n) \in (\mathbb{R}_+^2)^n$ and $r = [r_1, \dots, r_n] \in \mathbb{R}^n$ holds :

$$\sum_{i=1}^n \sum_{j=1}^n r_i r_j |z_i + z_j|^{-2} \geq 0. \quad (2.12)$$

In other words, the quadratic form :

$$r G_2(z) r^T := [r_1, \dots, r_n] [|z_i + z_j|^{-2}]_{n \times n} [r_1, \dots, r_n]^T,$$

is LHp in the **Sylvester sense**(see e.g. [Ko]).

(Let us mention on the danger of the fact that $|0|^{-2} = +\infty$). In this case we have deal with indefinite symbols : $\infty - \infty$.

Using the complex Laplace representation (1.3) we obtain (first we take elements from the semigroups $S_a = \mathbb{R}_a^2 := \{(x, y) \in \mathbb{R}^2 : x \geq a, y \geq a\}, a > 0$, i.e. $z \in S_a$)

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^n r_i r_j |z_i + z_j|^{-2} = \\ & = \sum_{i=1}^n \sum_{j=1}^n r_i r_j \int \int_{\mathbb{R}_+^2} e^{-\operatorname{re}(z_i + z_j)(l_1 + l_2)} \operatorname{re}(e^{i \operatorname{im}(z_i + z_j)(l_1 - l_2)}) dl_1 dl_2 = \\ & = \sum_{i=1}^n \sum_{j=1}^n r_i r_j \int \int_{\mathbb{R}_+^2} e^{-\operatorname{re}(z_i)(l_1 + l_2)} e^{-\operatorname{re}(z_j)(l_1 + l_2)} [\cos(\operatorname{im}(z_i)(l_1 - l_2)) \cos(\operatorname{im}(z_j)(l_1 - l_2)) - \\ & \quad - \sin(\operatorname{im}(z_i)(l_1 - l_2)) \sin(\operatorname{im}(z_j)(l_1 - l_2))] dl_1 dl_2 = \end{aligned} \quad (2.13)$$

$$= \int \int_{\mathbb{R}_+^2} \left(\sum_{i=1}^n \sum_{j=1}^n r_i \cos(\operatorname{im}(z_i)(l_1 - l_2)) r_j \cos(\operatorname{im}(z_j)(l_1 - l_2)) e^{-re(z_i)(l_1+l_2)} e^{-re(z_j)(l_1+l_2)} \right) dl_1 dl_2 \quad (2.14)$$

+

$$\int \int_{\mathbb{R}_+^2} \left(\sum_{k=1}^n \sum_{j=1}^n [ir_k \sin(\operatorname{im}(z_k)(l_1 - l_2))] [-ir_j \sin(\operatorname{im}(\tilde{z}_j)(l_1 - l_2))] e^{re(z_k)(l_1+l_2)} e^{-re(z_j)(l_1+l_2)} \right) dl_1 dl_2. \quad (2.15)$$

Let us put :

$$r'_j := r_j \cos(\operatorname{im}(z_j)(l_1 - l_2)) \in \mathbb{R}, \quad (2.16)$$

and

$$r''_j := ir_j \sin(\operatorname{im}(z_j)(l_1 - l_2)) \in \mathbb{C}, \quad (2.17)$$

$j = 1, \dots, n$. Then the first cosine double sum in (2.14) is equal to

$$\int \int_{\mathbb{R}_+^2} \left(\sum_{j=1}^n r'_j e^{-re(z_j)(l_1+l_2)} \right)^2 dl_1 dl_2 \geq 0, \quad (2.18)$$

whereas, the second sine double sum in (2.15) is equal to

$$\begin{aligned} \int \int_{\mathbb{R}_+^2} \left(\sum_{j=1}^n r''_j e^{-re(z_j)(l_1+l_2)} \right) c \left(\sum_{k=1}^n r''_k e^{-re(z_k)(l_1+l_2)} \right) dl_1 dl_2 &= \\ &= \int \int_{\mathbb{R}_+^2} \left| \sum_{j=1}^n r''_j e^{-re(z_j)(l_1+l_2)} \right|^2 dl_1 dl_2 \geq 0. \end{aligned} \quad (2.19)$$

Applying the Bernstein-Widder theorem for $|z|^{-2}$ on $\mathbb{R}_{1/n}^2$, we obtain the compatible sequence of **finite measures** $\{H_2^n\}$ with

$$|z|^{-2} = \int \int_{\mathbb{R}^2} e^{-z \cdot l} dH_2^l(l), \quad z \in \mathbb{R}_{1/n}^2. \quad (2.20)$$

The inductive limit of $\{H_2^n\}$ gives obviously the required measure H_2 .

Thus, we showed that $\operatorname{Rep}(\mathbb{C})$ immediately implies $\operatorname{Rep}(\mathbb{R})$, what is very exciting, since for many years, it seemed that it is not possible, and we always used the below Fernique-Haar measure systems.

Let us observe two "bad" properties of H_2 :

(i) $H_2(\mathbb{R}^2) = +\infty$, i.e. H_2 is an infinite measure,

and

(ii) the **support** $\operatorname{supp}(H_2)$ is \mathbb{R}_+^2 .

(Therefore, for many years- many maths people claimed that H_2 cannot exists! - by private communications) .

3 H_2 is produced by any Fernique-Haar system ^{*}.

This section can be omitted under the first reading of the paper, without any doubts for the understanding of this elementary analytic proof of the Riemann hypothesis.

According to the Bernstein-Widder theorem, we know that the measure H_2 **exists**, but we know nothing on the **construction** (or the inner structure) of the measure H_2 , i.e. our proof of existence of H_2 is not constructive - i.e. is "bad" - from the point of view of **Brouwer logic** and intuitionistic mathematics - like the famous Cantor's proof of the existence of transcendental numbers.

The below considered Fernique-Haar systems permit to **construct** H_2 and explain its inner structure.

Let A be any **measurable commutative algebra** with unit endowed with a σ -field \mathcal{A} of subsets of A and I be a **measurable linear subspace** of A . Let $\langle \cdot, \cdot \rangle : I \times A \rightarrow \mathbb{R}$ be a **Q-bilinear-measurable form** (w.r.t. \mathcal{A}) (see also [ML] and [MR] - for the different constructions of Riemann-Weil cohomologies).

Moreover, we also assume that is given a **way of immersion** of the vector space I into A via the **multiplication** on some non-zero element $i_0 \in I$, i.e. $M_{i_0} : I \rightarrow A$, where $M_{i_0}(i) := i \cdot i_0, i \in I$.

Moreover, we assume that there exists **I -invariant probability measure** $H : \mathcal{A} \rightarrow [0, 1]$ (in this paper - similarly like in [MR] - we call such measures the **weak (or residual) Haar measures**), since for each $i \in I$ and each $B \in \mathcal{A}$ we have :

$$(Inv) \quad H(B + i) = H(B).$$

Finally, we assume that there exists such a **positive constant** $f = f(A) > 0$ that for each $m \in (0, f)$ the integral

$$(FC) \quad F_H^m(A) := \int_A e^{\frac{m \langle a^2, 1 \rangle}{2}} dH(a)$$

is **finite**. From (FC) we immediately obtain the existence of a family of **finite Feynmann-Kac measures** F_H^m , defined as :

$$F_H^m(B) := \int_B e^{m \langle a^2, 1 \rangle / 2} dH(a), \quad B \in \mathcal{A}. \quad (3.21)$$

In the sequel such systems (sixtets)

$$(A, \mathcal{A}, I, \langle \cdot, \cdot \rangle, H, f(A)),$$

we call the **Fernique-Haar systems**. In the proof of the Riemann hypothesis such systems plays such a good role like the famous **Kolygavin-Euler systems** in the proof of the Fermat Last Theorem.

Let $\mathbb{A} = (A, \mathcal{A}, \langle \cdot, \cdot \rangle, I, H, f(A))$ be any Fernique-Haar system. Let $m \in (0, f(A))$ and let

$$dF_m(a) = e^{\frac{m \langle a, a \rangle}{2}} dH(a),$$

be the **finite Feynmann-Kac measure**. Let us calculate the **Laplace transform** $\mathcal{L}_I(F_m)$ of F_m (w.r.t. the \mathbb{R} -linear-biform $m < \cdot, \cdot >$ and restricted to I):

$$\begin{aligned}\mathcal{L}_I(F_m)(i) &:= \int_A e^{m\langle ia, 1 \rangle} dF_m(a) = \int_A e^{m\langle ia, 1 \rangle} e^{\frac{m\langle a^2, 1 \rangle}{2}} dH(a) = \\ &= e^{-\frac{m\langle i^2, 1 \rangle}{2}} \int_A e^{(m/2)(\langle i^2, 1 \rangle + 2\langle ia, 1 \rangle + \langle a^2, 1 \rangle)} dH(a) = e^{-m\langle i, i \rangle/2} \int_A e^{\frac{m\langle (i+a)^2, 1 \rangle}{2}} dH(a) = \\ &= e^{-m\langle i^2, 1 \rangle/2} \int_A e^{m\langle a^2, 1 \rangle/2} dH(a) = e^{-\frac{m\langle i^2, 1 \rangle}{2}} F_m(A),\end{aligned}$$

since, obviously H is **I -invariant** and the biform $< \cdot, \cdot >$ is \mathbb{Q} -linear. (Let us mention at this place, that opposite to the von Neumann -Weil theorem, can exists diferent (up to a constant) I -invariant measures if $I \neq A$).

Let us fix any $i_0 \in I \neq \{0\}$. Then according to the above calculations, for any $u, v \in \mathbb{Q}$ we have

$$\begin{aligned}\mathcal{L}_I(F_m)(ui_0)\mathcal{L}_I(F_m)(vi_0) &= e^{-m\langle (u^2+v^2)i_0^2, 1 \rangle} F_m^2(A) = \\ &= \int_{A^2} e^{m(u\langle i_0, a_1 \rangle + v\langle i_0, a_2 \rangle)} d(F_m \otimes F_m)(a_1, a_2).\end{aligned}\tag{3.22}$$

Since - obviously - \mathbb{Q} is **dense** in \mathbb{R} , then the formula (3.22) holds for all $(u, v) \in \mathbb{R}^2$.

Integrating (3.22) with respect to the **Lebesgue measure** dt on \mathbb{R}_+ we get

$$\begin{aligned}\frac{F_m^2(A)}{m(u^2+v^2)\langle i_0^2, 1 \rangle} &= F_m^2(A) \int_{\mathbb{R}_+} e^{-[(u^2+v^2)m\langle i_0^2, 1 \rangle]t} dt = \\ &= F_m^2(A) \int \int_{\mathbb{R}_+ \times A^2} e^{[m\sqrt{t}(u\langle i_0, a_1 \rangle + v\langle i_0, a_2 \rangle)]} dt \otimes d(F_m \otimes F_m)(a_1, a_2).\end{aligned}\tag{3.23}$$

Let us consider the measure space $(\mathbb{R}_+ \times A^2, dt \otimes dF_m^2)$ and the random variable

$$X_{mi_0}(t, a_1, a_2) = m\sqrt{t}(\langle i_0, a_1 \rangle, \langle i_0, a_2 \rangle) \in \mathbb{R}^2.$$

Let us put :

$$H_2^* := \frac{m\langle i_0^2, 1 \rangle}{F_m^2(A)} X_{mi_0}^*(F_m^2 \otimes dt).\tag{3.24}$$

Using the transport of measure theorem, we finally get : $z = (u, v) \in \mathbb{C}$

$$\begin{aligned}|z|^{-2} &= \int \int_{\mathbb{R}_+^2} e^{ux_1+vx_2} dX_{mi_0}^*(F_m^2 \otimes dt)(x_1, x_2) = \\ &= \int_{\mathbb{R}_+^2} e^{-z \cdot l} dH_2^*(l),\end{aligned}\tag{3.25}$$

since $\text{supp}(H_2) = \mathbb{R}_-$. According to the **uniqueness** of a Laplace representation, we finally get : $H_2^* = H_2$, i.e. H_2^* is a required Hodge measure and moreover is produced by our Fernique-Haar system.

Remark 3 Our Fernique-Haar systems introduced and considered here are modeled on the following **Fernique-Girsanov system** considered in [ML] :

$$(C_0, \mathcal{C}, \mathbb{R} \cdot L_a, <, >_{L^2}, w_\infty, f(w_\infty)),$$

where C_0 is the Frechet space of all continuous functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, with $f(0) = 0$, L_a is a peak function, the bilinear form $<, >_{L^2}$ is some difference of Hilbert space L^2 -scalar product (with two different measures), w_∞ is the **standard Wiener measure** and $f(w_\infty)$ its Fernique's constant of w_∞ .

Obviously w_∞ is not \mathbb{R} -invariant but only \mathbb{R} -quasi-invariant, since according to the **Girsanov theorem** we have

$$w_\infty(B + rL_a) = \int_B e^{-\frac{1}{2}r^2 \int_0^\infty (L'_a(t))^2 dt - r \int_0^\infty L'_a(t) dx(t)} dw_\infty(x),$$

where $r \in \mathbb{R}$ and B is a Borel set in C_0 .

Let C be the Frechet space of all real-valued continuous functions on \mathbb{R}_+ . Then obviously we have the decomposition : $C = C_0 \oplus \mathbb{R}$ and there exists a natural extension of w_∞ to C , being the distribution of the whole family of **Brownian motions** $B_a = (B_a(t) : t \geq 0)$ starting from all points $a \in \mathbb{R}$. Let $B = (B_a(t) : t \geq 0, a \in \mathbb{R})$. Let $W_\infty := \text{Law}(B)$ be the **distribution (law)** of the stochastic field B . Unfortunately, W_∞ is not \mathbb{R} -invariant, since for each $a \in \mathbb{R}$ the measure $W_\infty(\cdot + a)$ is **singular** to W_∞ . Really, the distribution of a r.v. $B_a(0)$ is the Dirac delta point measure δ_a , i.e. $\text{Law}(B_a(0)) = \delta_a$ and $\text{supp}W_\infty(\cdot + a) \cap \text{supp}W_\infty = O$.

But, we can easily delete that disadvantage, giving the following easy \mathbb{R} -invariant extension of w_∞ to C : since obviously $C_0 \oplus \mathbb{R} \simeq C_0 \times \mathbb{R}$ then we can define the product measure \mathcal{W}_∞ by the formula :

$$\mathcal{W}_\infty := w_\infty \otimes l,$$

where l is the Lebesgue measure on $(\mathbb{R}, +)$. Then obviously \mathcal{W}_∞ is \mathbb{R} -invariant but **infinite** and does not satisfy the Fernique condition (FC).

But it suffices to take the Frechet group $C(\mathbb{R}_+, \mathbb{T})$ of all continuous functions on \mathbb{R}_+ with values in the 1-dimensional torus \mathbb{T} (instead of \mathbb{R}) and a Haar probability measure $H_{\mathbb{T}}$ on its, to obtain a proper measure $W_\infty := w_\infty \otimes H_{\mathbb{T}}$.

Example 1 (The l-adic Wiener-Fernique-Haar systems.)

Let $C_l = C(\mathbb{Z}_l, \mathbb{Q}_l)$ be the l-adic Banach space of all \mathbb{Q}_l -valued continuous functions defined on \mathbb{Z}_l (with the sup-norm). Let \mathcal{B}_l be its Borel σ -field. In [MR] we constructed (combinatorially) the non-trivial \mathbb{Q} -linear homomorphism $I_l \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}_l^+, \mathbb{R}^+) = L_{\mathbb{Q}}(\mathbb{Q}_l^+, \mathbb{R}^+)$. If $f, g \in C_l$, then we define the biform $< \cdot, \cdot >_l$ by the formula :

$$< f, g >_l := I_l(f(1)g(1)).$$

Let w_l be the standard l-adic Wiener measure on \mathcal{B}_l (see [MR]). Then w_l has a compact \mathbb{Z}_l -module support : $K_l = \text{supp}(w_l)$ and is K_l -invariant. Then the system :

$$(C_l, \mathcal{B}_l, < \cdot, \cdot >_l, K_l, w_l, +\infty),$$

is a Fernique-Haar system. In particular we have got the l -adic Feynmann-Kac measure F_l given by the formula :

$$F_l(B) = \int_{B \cap K_l} e^{I_l(x^2(1))} dw_l(x), \quad B \in \mathcal{B}_l,$$

(which do not depend on any parameter m). Since w_l is a **Haar measure** on K_l (see [Mqdrecki, PhD. Thesis]) and $I_l : \mathbb{Q}_l \longrightarrow \mathbb{R}$ is **continuous**, then the Fernique lemma is trivially satisfies in this case - and - $f(C_l) = f(w_l) = +\infty$.

It is easy to see that the l -adic Laplace transform $\mathcal{L}(F_l)$ has the form :

$$\mathcal{L}(F_l)(x) = e^{-I_l(x^2(1))/2} F_l(C_l).$$

Example 2 (The l -adic Gibbs-Fernique-Haar systems).

Let \mathbb{Q}_l be the l -adic number fiel, b_l its Borel σ -algebra and $>., <_l$ the bilinear form defined by the formula :

$$> x, y <_l := I_l(xy), \quad x, y \in \mathbb{Q}_l.$$

Finally, let H_l be the Haar measure of \mathbb{Q}_l normalized by $H_l(\mathbb{Z}_l) = 1$. The the systems

$$(\mathbb{Q}_l, b_l, >., <_l, l^n \mathbb{Z}_l, H_l, +\infty),$$

are the Fernique-Haar systems. We can define the l -adic Gibbs-F-K -measures according to the formula :

$$f_l(B) := \lim_n \int_{B \in l^n \mathbb{Z}_l} e^{I_l(x^2)/2} dH_l(x),$$

for **bounded** set $B \subset l^m \mathbb{Z}_l$ (it is so called l -adic thermodynamical limit) and obviously there is the formula for suitable l -adic (one-dimensional) Laplace transform of the form :

$$L(f_l)(x) = e^{-I_l(x^2)/2} f_l(l^m \mathbb{Z}_l).$$

Example 3 The measurable cardinals-Pelc-Fernique-Haar systems.

The notion of a measure was introduced to mathematics at the begining of the twenty century in connection of some problems of real functions theory. The measure theory leaded to some purely set theoretic problems. One of such branches of the modern set theory is the theory of **measurable cardinals**.

In the modern set theory is also used the following extended notion of a measure (see [K,Sect.10.6]).

If X is a set and $f : X \longrightarrow \mathbb{R}_+$ a positive function, then by $\sum_{x \in X} f(x)$ is denoted the supremum of the set of real numbers of the form $\sum_{x \in F} f(x)$, where F is any finite subset of X . In the case $X = \mathbb{N}$ the sum $\sum_{x \in X} f(x)$ is equal to $\lim_n s_n$, where $s_n = \sum_{i < n} f(i)$.

For a set X , $|X|$ denotes the **cardinality** of X and $P(X)$ the family of all subsets of X . Let κ be any **cardinal**. We propose the following definition :

Definition 3.1 Let G be any **discrete abelian group** and $I \triangleleft G$ a non-zero subgroup (Let us mention that any topological group can be considered with its **discrete topology** G_d - see e.g. [Hartman]). By a **Pelc measure** p , we understand here any set function which satisfies the following four conditions :

(P_1) $p : P(G) \longrightarrow [0, 1], p(O) = 0, p(G) = 1$, i.e. p is a **probability universal measure**.

(P_2) p is **κ -additive**, i.e. $p(\sum X) = \sum_{X \in \mathcal{X}} p(X)$ for each family $\mathcal{X} \subset P(G)$ of commonly disjoint sets and such that $|\mathcal{X}| < \kappa$. Here we consider only σ -additive measures (see [P]), i.e. $\kappa = \omega = |\mathbb{N}|$.

(P_3) $p(\{g\}) = 0$ for each $g \in G$, i.e. p **vanishes on singletons**.

(P_4) p is **I -invariant**, i.e. for each $i \in I$ and any subset S of G there is :

$$p(S + i) = p(S).$$

Let us recal that a cardinal κ is called a **measurable cardinal** (mc for short), if on some set of power κ there exists a universal probability measure (see [K] and [P, 0.1]).

Many classes of measures and different types of measurable cardinals - e.g. real-valued and **Ulam cardinals** were considered in [P], where in particular, were established some beautiful theorems on the existence of Pelc measures.

In this example we say shortly measurable cardinal (mc) like in [K] and not like in [P], where is considered "the whole ZOO" of mc's.

As it is well-known, according to the famous **Banach-Kuratowski theorem** (see [BK]) : "if $2^\omega = \omega_1$ then 2^ω is not (mc)" and **Ulam theorem** (see [P, Th.1.5]) - the **general measure problem**, i.e. the problem : does there exist a σ -additive σ -finite measure vanishing on singletons defined on all subsets of the reals ? - cannot have a positive solution in usual mathematics. In this setting the problem becomes purely set theoretic, i.e. depends exclusively on the cardinality of the underlying set.

More exactly, assuming the **Continuum hypothesis** (Ch for short), such a measure cannot exist. But - as showed **Cohn with Godel** - Ch is **independent** from the axioms of ZFC (see [K]). Nevertheless the possibility of disproving the existence of (weakly) **inaccessible cardinals** or real valued and Ulam or measurable cardinals is not excluded. In spite of this danger the assumptions that such cardinals exist are commonly used in modern set theory as **additional axioms** (see [P] and [K]), since :

1⁰. According to **Solovay theorem**, the following statements are equivalent :

(S_1) " (mc) exists" is consistent with ZFC

and

(S_2) The statement " 2^ω is (mc)" is consistent with ZFC,

as well as - according to the **Kunen theorem**, the following are equivalent :

(K_1) The statement "mc exists" is consistent with ZFC ,

(K_2) The statement " There exists a κ -complete λ -saturated ideal on a cardinal κ " (where $\omega_1 \leq \lambda \leq \kappa^+$) is consistent with ZFC (see [P]).

Let us note that the positive answer to the original version of the general measure problem (gmp), i.e. the statement " 2^ω is mc " is in a sense strictly stronger than the

statement "there exists mc ". **Levy** and **Solovay** have proved that if the existence of (mc) is consistent with set theory then the statement "there exists a $mc + 2^\omega = \omega_1$ " is also consistent with ZFC and hence in view of the above mentioned result of **Banach** and **Kuratowski** the statement "there exists a $(mc) + 2^\omega$ is not (mc) " is consistent.

Finally a remark should be made about that set theoretic framework. All the mentioned results are proved in usual set theory with the axiom of choice(ZFC).

Similarly like (gmp) cannot be solved positively in the usual set category, the **invariant** and **weak-invariant** gmp cannot be positively solved in the usual (ZFC) category of **groups**, according to two well-known **deep negative results** (see [P]) of **Harazišvili-Erdos-Mauldin** and a more general **Ryll- Nardzewski-Telgarsky's result**(see [RNT]) : if I is any **uncountable subgroup** of a group G , there does not exists a **universal I -invariant σ -additive measure** on G , i.e. - in our terminology a **Pelc measure**.

Let A be a commutative algebra with unit and endowed with such a \mathbb{Q} - bilinear form $\langle ., . \rangle$ that it is **exponentially square bounded** on $A \times A$, i.e.

$$\sup_{a \in A} e^{\langle a, a \rangle} < +\infty.$$

Such algebras obviously exist - for example - we can take the compact ring \mathbb{Z}_l for A and the continuous biform $I_l(xy)$ for $\langle x, y \rangle$.

Then obviously the **Fernique Condition(FC)** is trivially satisfies for any **Pelc measure** $p : P(A) \longrightarrow [0, 1]$, i.e. any non-trivial universal probability and A -invariant measure p (it is convenient to call such measures the **Haar-Pelc measures**), since

$$F_p(A) = \int_A e^{\frac{\langle a^2, 1 \rangle}{2}} dp(a) \leq \sup_{a \in A} e^{\langle a^2, 1 \rangle / 2} < +\infty. \quad (3.26)$$

Let a **cardinal** α be the power of A , i.e. $\alpha = |A|$. Assume that α is a **measurable cardinal** (mc). Then according to the **Pelc theorem**[P, Th.2.5] - there exists a **Haar-Pelc measure** p on A . Thus, we see that we have also to our disposals the following **measurable cardinals Pelc-Fernique systems**:

$$(A, P(A), \langle ., . \rangle, A = I, p, +\infty =: f(A)). \quad (3.27)$$

The existence of systems (3.27) leads to the following - extremaly surprising connection between the **Continuum hypothesis**(Ch for short) and the **Riemann hypothesis**(Rh for short) given in the following logical implication (let us call it the **Measurable Riemann hypothesis**(MRh for short)):

(MRh) **The existence of measurable cardinals implies the Riemann hypothesis**, i.e. $(mc) \implies (Rh)$ (in short) (see also for (mRh) in [ML] and the last remark of this paper).

Now, doing the above general simple quadratic calculus in this case of measurable Pelc-Fernique-Haar systems we get

$$\mathcal{L}_A(p)(a) = e^{-\langle a^2 / 2, 1 \rangle} F_p(A),$$

i.e. we obtain the **bounded Gaussian density** from the **unbounded** one.

Finally, let us remark, that if G is a compact abelian group then there exists the probability Haar measure on it, i.e. the unique G -invariant probability measure H_G defined on a very small σ -algebra of **Borel sets** $\mathcal{B}(G)$ of G . In particular $p \mid \mathcal{B}(A) = H(A, +)$ is the probability Haar measure if A is compact.

If we take the **Wiener measure** w_∞ on the (non-locally compact) group $G = C(\mathbb{R}_+)$ of all continuous real valued function on \mathbb{R}_+ , then obviously $w_\infty(G) = 1$ and obviously $p_{C(\mathbb{R}_+)} \neq w_\infty$, since w_∞ is not the Haar (or at least \mathbb{R} -invariant).

4 An elementary proof of the Riemann hypothesis.

Theorem 3 (The Riemann hypothesis).

Let $\zeta = \zeta(s)$ be the **Riemann zeta function**. Then

$$(Rh) \quad \text{If } \zeta(s) = 0 \text{ and } \text{im}(s) \neq 0 \text{ then } \text{re}(s) = \frac{1}{2}.$$

Proof. In this proof we will use the following tools from the classical analysis :

1. The Riemann analytic continuation equation (Race for short).
2. The convergence of Dirichlet series.
3. The Newton-Leibnitz formula.
4. The change of summation and integration, i.e. the Fubini theorem for series.
5. The $Rep(\mathbb{R})$ as well as $Rep(\mathbb{C})$ representations of the Green function $|z|^{-2}$.
6. The positivity of Fresnel integrals.

Thus - in some sense - our proof is based on the shoulders of maths giants.

(P_1)(**Elementary analicityties and Rhfe- from Race to Rhfe**).

According to the **classical Riemann analytic functional continuation equation** (cf.e.g. [KV, L.2]) we have

$$\begin{aligned} i(s) &:= \text{Im}(\pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s)) =: \text{Im}(\zeta^*(s)) = \\ &= \text{Im}(\frac{1}{s(s-1)}) + \text{Im}(\int_1^\infty (x^{\frac{s-2}{2}} + x^{-\frac{(s+1)}{2}}) \theta(x) dx) \end{aligned} \quad (4.28)$$

, where all in the sequel by $G(x) = e^{-\pi x^2}$ we mean the **canonical Gaussian function** and $\theta(x)$ stands for the **canonical Jacobi theta**:

$$\theta(x) := \sum_{n=1}^\infty e^{-\pi n^2 x} = \sum_{n=1}^\infty G(n\sqrt{x}). \quad (4.29)$$

Let us denote :

$$J(s) := \int_1^\infty (x^{\frac{(s-2)}{2}} + x^{-\frac{(s+1)}{2}}) \theta(x) dx, \quad (4.30)$$

and

$$2J_n(x) := \int_1^\infty (x^{\frac{s-2}{2}} + x^{-\frac{s+1}{2}}) G(n\sqrt{x}) dx. \quad (4.31)$$

Let us observe that we can remove the fractions from the above subintegral expression if we change variables according to the substitution : $x^2 = t, dx = 2t dt$. Then

$$2J_n(s) = 2 \int_1^\infty (t^{s-1} + t^{-s}) G(nt) dt. \quad (4.32)$$

Now, let us observe that we have the following approximation of $|J(s)|$:

$$\begin{aligned} \sum_{n=1}^\infty \int_1^\infty (x^{u-1} + x^{-u}) G(nx) dx &\leq \|G\|_{(m,0)} (:= \sup_{x \in \mathbb{R}} |G(x)x^m|) \times \\ \sum_{n=1}^\infty \int_1^\infty \frac{|x^{u-1} + x^{-u}|}{n^m x^m} dx &\leq 2 \|G\|_{(m,0)} \zeta(m) \max_{1 \leq x \leq \infty} (x^{u-1}, x^{-u}) \int_1^\infty \frac{dx}{x^m} < \infty, \end{aligned} \quad (4.33)$$

if $u = \operatorname{Re}(s) \in I := [0, 1]$ (in fact if $u \leq 1$) and $m > 1$.

Thus (4.33) shows that the **iterated integral**

$$\sum_{n=1}^\infty \int_1^\infty |x^{u-1} - x^{-u}| G(nx) dx$$

is **finite** if $m > 1$ and $u = \operatorname{Re}(s) \leq 1$.

Since the **Lebesgue measure** dt on $(\mathbb{R}, +)$ and the **calculating measure** dc on $(\mathbb{Z}, +)$ are σ -**finite**, then according to the **Fubini theorem** for positive functions (cf.e.g. [B, Commentaries after Th.18.2 and Exercise 18.3]), the functions $f_s(x, n) := |x^{\operatorname{Re}(s)-1} - x^{-\operatorname{Re}(s)}| G(nx)$ are **flat-integrable** with respect to the product measure $dt \otimes dc$ on $[1, +\infty) \times \mathbb{N}$. Then according to the Fubini theorem for arbitrary integrable functions (i.e. the **Tonelli theorem**) we have

$$\begin{aligned} \operatorname{Im}(J(s)) - 2 \int_1^\infty (x^{\operatorname{Re}(s)-1} - x^{-\operatorname{Re}(s)}) \sin(\operatorname{Im}(s) \log x) \theta(x) dx &= \\ &= 2 \sum_{n=1}^\infty \operatorname{Im}(J_n(s)). \end{aligned} \quad (4.34)$$

The **detailed calculation** of $\operatorname{Im}(J_n(s))$ for s with $\operatorname{Re}(s) \in (0, 1)$ is a "heart" of this "complex Laplace representation proof of (Rh)" - in the spirit of the **classical analytic number theory**.

That calculations and the effect of them was quite surprising for us many years ago!

Thus, for each $n \in \mathbb{N}$ and $s = u + iv$ with $\operatorname{Re}(s) \in (0, 1)$ and $v = \operatorname{Im}(s) \in \mathbb{R}^*$ we have

$$\begin{aligned} \operatorname{Im}(J_n(s)) &= \int_1^\infty (x^{u-1} - x^{-u}) \sin(v \log x) G(nx) dx = \\ &= \int_0^\infty (e^{r(u-1)} - e^{-ru}) \sin(vr) G(ne^r) e^r dr, \end{aligned} \quad (4.35)$$

if we apply the **change variables formula**: $x = e^r, dx = e^r dr$.

But

$$\begin{aligned} & \int_0^\infty (e^{ru} - e^{r(1-u)}) \sin(vr) e^{-\pi n^2 e^{2r}} dr = \\ &= \lim_{N \rightarrow \infty} \int_0^N \left(\sum_{j=0}^\infty \frac{(-1)^j (\pi n^2 e^{2r})^j}{j!} \right) (e^{ru} - e^{r(1-u)}) \sin(vr) dr. \end{aligned} \quad (4.36)$$

Since the **Taylor expansion** of G is uniformly convergent on each closed segment of \mathbb{R} , then

$$\begin{aligned} & \int_0^N (e^{ru} - e^{r(1-u)}) \sin(vr) \left(\sum_{j=0}^\infty \frac{(-\pi n^2 e^{2r})^j}{j!} \right) dr = \\ & \sum_{j=0}^\infty \frac{(-\pi n^2)^j}{j!} \int_0^N (e^{r(2j+u)} - e^{r(2j+1-u)}) \sin(vr) dr. \end{aligned} \quad (4.37)$$

But, it is an elementary fact that the following defined integrals are easy calculable by the **Newton-Leibnitz formula**

$$\begin{aligned} & \int_0^N e^{wr} \sin(vr) dr = \frac{e^{wr} (w \sin(vr) - v \cos(vr))}{w^2 + v^2} \Big|_0^N = \\ & e^{Nw} \frac{(w \sin(vN) - v \cos(vN))}{w^2 + v^2} + \frac{v}{w^2 + v^2}. \end{aligned} \quad (4.38)$$

Since always (for a fixed $v = \text{Im}(s) \neq 0$) we can choose the sequence in (4.37) and (4.38) of the form :

$$N := \frac{2\pi L}{v}, \quad L \in \mathbb{N}, \quad (4.39)$$

then the formula (4.38) will obtain a simpler form

$$\int_0^{2\pi L/v} e^{wr} \sin(vr) dr = \frac{v}{v^2 + w^2} (1 - e^{2\pi Lw/v}), \quad L \in \mathbb{N}. \quad (4.40)$$

(P_2)(**The vanishing of the Poissonian part $P_n(s)$ according to $\text{Rep}(\mathbb{C})$**).

Now, it will be convenient to do the following digression and notation : let a and b be arbitrary real numbers. By P_a^b we denote a **generalized and signed Poisson** random variable with the parameters a, b . Thus

$$\text{mes}(P_a^b = k) = e^{-a} \cdot \frac{b^k}{k!}, \quad k \in \mathbb{N}.$$

If $a = b = \lambda > 0$, then we obtain the **Poisson distribution** with a parameter λ .

Let $p_a^b(x) = \text{mes}(P_a^b \leq x)$ be the generalized and signed **Poisson distribution function**. Then its **Escher characteristic function \hat{P}_a^b** (being nothing that the **real two-sided Laplace transform with the unbounded exponential kernel**) has the form :

$$\hat{P}_a^b(u) := \int_0^\infty e^{ux} dp_a^b(x) = e^{-a} \sum_{k=0}^\infty \frac{e^{uk} b^k}{k!} = e^{-a+be^u}, \quad u \in \mathbb{R}.$$

Combining the identities from (4.34) to (4.40) we obtain

$$\begin{aligned}
Im(J_n(s)) &= \lim_{L \rightarrow \infty} \sum_{j=0}^{\infty} \frac{(-\pi n^2)^j v}{j!} (e^{\frac{2\pi L(2j+1-u)}{v((2j+1-u)^2+v^2)}} - e^{\frac{2\pi L(2j+u)}{v(2j+u)^2+v^2}}) + \\
&\sum_{j=0}^{\infty} \frac{(-\pi n^2)^j}{j!} \cdot \frac{v(2u-1)(4j+1)}{|(s+2j)(s-2j-1)|^2} =: \\
&=: P_n(s) + \zeta_t(s) Tr_{CG}^n(s).
\end{aligned} \tag{4.41}$$

In particular, we see that the limit $Tr_{CG}^n(s)$ exists!

Additionally, we remark that

$$Im\left(\frac{1}{s(s-1)}\right) = \frac{\zeta_t(s)}{|s(s-1)|^2} =: \zeta_t(s)t_0(s), \tag{4.42}$$

i.e. the **zero-polar term** $\frac{1}{s(s-1)}$ of $\zeta(s)$ is entered into the consideration of the below trace sequence $\{t_n(s)\}$.

Now, using the complex Laplace representation $Rep(\mathbb{C})$ of $|z|^{-2}$, we show that for each $n \in \mathbb{N}^*$, the **Poissonian term** $P_n(s)$ **vanishes**, i.e.

$$P_n(s) = 0 \text{ for } n \in \mathbb{N}^* \text{ and } s \text{ with } Re(s) \in I. \tag{4.43}$$

To do that let us denote

$$\begin{aligned}
P_n^0(z) &:= \lim_L \sum_{j=0}^{\infty} \frac{e^{\frac{2\pi L(2j+Re(z))}{Im(z)}}}{|2j+z|^2} \frac{(-\pi n^2)^j}{j!} =: \\
&=: \lim_L \left(\sum_{j=0}^{\infty} e^{a(z)L} \frac{(e^{b(z)L})^j}{|2j+z|^2} \frac{(-\pi n^2)^j}{j!} \right),
\end{aligned} \tag{4.44}$$

where $z = s$ or $z = 1-s$, $u = Re(s)$, $v = Im(s) \neq 0$ and $j, L \in \mathbb{N}$. Finally, we have introduced also notations :

$$a(z) := \frac{2\pi Re(z)}{v} > 0 \text{ and } b(z) := \frac{4\pi}{v} > 0.$$

According to (1.3) and (1.4) we have

$$\frac{1}{|2j+z|^2} = \int_{\mathbb{R}_+^2} e^{-\langle 2j+z, l \rangle} d^2 l. \tag{4.45}$$

Therefore, we have ($z = s$ or $z = 1-s$)

$$P_n^0(z) = \lim_L \left(e^{a(z)L} \left(\sum_{j=0}^{\infty} \frac{(-\pi n^2)^j}{j!} \right) (e^{b(z)L})^j \right) \int_{\mathbb{R}_+^2} e^{-\langle 2j+z, l \rangle} d^2 l = \tag{4.46}$$

$$\begin{aligned} \lim_L (e^{a(z)L} \int \int \mathbb{R}_+^2 (\sum_{j=0}^{\infty} \frac{(-\pi n^2 e^{b(z)L})}{j!} e^{-2(l_1+l_2)})^j) e^{-\langle z, (l_1, l_2) \rangle} dl_1 dl_2 = \\ = \lim_L (e^{a(z)} \int \int_{\mathbb{R}_+^2} e^{-\pi n^2 e^{(b(z)L - 2(l_1+l_2))}} e^{-\langle z, l \rangle} dl_1 dl_2. \end{aligned}$$

The limit transition in the second line of the above three lines (4.46) is become valid (under fixed (n, L, z)), since the assumptions of the **Tonelli theorem** for the **space measure** $dc \otimes dl_1 \otimes dl_2$ are satisfies :

$$\begin{aligned} \sum_{j=0}^{\infty} \int \int_{\mathbb{R}_+^2} \frac{(\pi n^2 e^{b(z)L})^j}{j!} e^{-2(j+Re(z))(l_1+l_2)} dl_1 dl_2 = \sum_{j=0}^{\infty} \frac{(\pi n^2 e^{b(z)L})^j}{j!} \cdot \frac{1}{4(j+Re(z))^2} \leq \\ \leq (4sup_j (j+Re(z))^{-2}) e^{\pi n^2 e^{b(z)L}} < +\infty. \end{aligned}$$

Finally, introducing the last notation : $c_n = \pi n^2$, we can write down $P_n(z)$ as the limit :

$$P_n^0(z) := \lim_L P_n^0(L, z), \quad (4.47)$$

where

$$P_n^0(L, z) := \int \int_{\mathbb{R}_+^2} e^{-c_n e^{b(z)L - 2(l_1+l_2) + a(z)}} e^{-Re(z)(l_1+l_2)} dl_1 dl_2. \quad (4.48)$$

But using a very coarse approximation : $e^x \geq \frac{x^r}{r!}, r \in \mathbb{N}^*$, we obtain (in short)

$$\begin{aligned} 0 \leq e^{aL - c_n e^{(bL - 2(l_1+l_2))}} &\leq \frac{r! e^{aL} e^{2r(l_1+l_2)}}{c_n^r e^{rbL}} \leq \\ &\leq \frac{r!}{\pi^r n^{2r}} e^{(a-br)L} e^{2r(l_1+l_2)}. \end{aligned} \quad (4.49)$$

Therefore

$$\begin{aligned} 0 \leq |P_n^0(L, z)| \leq \frac{r!}{\pi^r n^{2r}} \int \int_{\mathbb{R}_+^2} e^{\frac{2\pi}{Im(z)}(Re(z)-2r)L} e^{2r(l_1+l_2)} |e^{-\langle z, (l_1, l_2) \rangle}| dl_1 dl_2 = \\ = \frac{r! e^{\frac{2\pi}{Im(z)}(re(z)-2r)L}}{\pi^r n^{2r}} (\int \int_{\mathbb{R}_+^2} e^{-(re(z)-2r)(l_1+l_2)} dl_1 dl_2 = (re(z) - 2r)^{-2}) =: p_L(z, n, r). \end{aligned} \quad (4.50)$$

Under a fixed z with $re(z) \in (0, 1)$ and $n \in \mathbb{N}^*$, taking $r \geq 1$, we obtain

$$0 \leq |P_n^0(z)| \leq \lim_L p_L(z, n, r) = 0. \quad (4.51)$$

To finish the proof of this part of the proof, it suffices to observe that : $P_n(s) = P_n^0(s) - P_n^0(1-s) = 0$.

(P₃)(**There existence of a moment representation of the trace sequence $\{t_j(s)\}$**).

Before we start to prove the subsequence part of the proof, we introduce the **trace sequence** $t(s) = \{t_j(s)\}$ of ζ by the following formula :

$$t_j(s) := \frac{4j+1}{|(s+2j)(2j+1-s)|^2} , j \in \mathbb{N}. \quad (4.52)$$

Now, we remark, that the complex Laplace transform representation $Rep(\mathbb{C})$ of $|z|^{-2}$, is **too weak** to obtain the subsequence part of the elementary analytic proof of the Riemann hypothesis. We will base very strongly on there existence of the **Hodge measure** H_2 and respectable **real Laplace representation** $Rep(\mathbb{R})$. We prove that it induces the following : for each $s \in \mathbb{C}$ with $re(s) \in (1/2, 1)$ and $im(s) < 0$ holds :

$$t_j(s) = \int_0^1 x^j dh_s(x) , j = 0, 1, 2, \dots, \quad (4.53)$$

i.e. $t_j(s)$ is the j -**th moment** of the **Hodge measure** h_s . Realy, let us write down the sequence $t(s) = \{t_j(s)\}$ of the form

$$t_j(s) = \frac{1}{| \frac{(0.5-s)}{4j+1} + 0.5 |^2} \cdot \frac{1}{4j+1} \cdot \frac{1}{|s+2j|^2}. \quad (4.54)$$

In such a way, the decomposition of $t_j(s)$ is the main point of this elementary proof of Rh.

The most difficult and fundamental in fact, is the possibility of a real Laplace representation of the **first factor** in the above product, and therefore we first start from a consideration of that factor. According to (2.20) , we have

$$|z|^{-2} = \int \int_{\mathbb{R}_+^2} e^{-z \cdot l} dH_2(l) , z \in \mathbb{R}_+^2. \quad (4.55)$$

Hence

$$\begin{aligned} \frac{1}{| (0.5-s)/(4j+1) + 0.5 |^2} &= \int \int_{\mathbb{R}_+^2} e^{-((0.5-s)/(4j+1)) \cdot l} e^{-0.5 \cdot l} dH_2(l) = \\ &= \int \int_{\mathbb{R}_+^2} e^{\frac{-(0.5-re(s)) \cdot l_1}{4j+1}} e^{\frac{-im(s) \cdot l_2}{4j+1}} e^{-0.5 \cdot l_1} dH_2(l_1, l_2) = \\ &= \int \int_{\mathbb{R}_+^2} \left(\sum_{n=0}^{\infty} \frac{((re(s) - 0.5)l_1)^n}{n!(4j+1)^n} \right) \left(\sum_{k=0}^{\infty} \frac{(-im(s)l_2)^k}{k!(4j+1)^k} k!(4j+1)^k \right) e^{-0.5l_1} dH_2(l_1, l_2) = \end{aligned} \quad (4.56)$$

(Let us observe that all here , i.e. in this - the last but one part - of the proof of the Riemann hypothesis concerning the existence of the family of trace Hodge measures $\{h_s\}$, we assume that $re(s) \in (1/2, 1]$ and $im(s) < 0$, since it guarantees the **integrability** of the subintegral function w.r.t. H_2).

But according to the **Bernstein-Hausdorff theorem** (see [F., VII.9, Exercice 6 and XIII.4, Th.1 and XIII.1, Example(b)]), we have

$$r^{-l} = \int_0^\infty e^{-ru} \left(\frac{u^{l-1} du}{(l-1)!} =: dB_l(u) \right), l \geq 1, l \in \mathbb{N}, \quad (4.57)$$

, where B_l is the "**Bernstein measure**". (Let us observe that in this place we have got a problem : the value $\Gamma(0)$ cannot be represented by the integral $\int_0^\infty u^{l-1} e^{-ru} du$, for $l = 0$, since it last is divergent!).

Therefore, the integral in (4.56) we can write of the form :

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \left(\sum_{k,n=0}^\infty \frac{((re(s) - 0.5)l_1)^n (-im(s)l_2)^k e^{l_1/2}}{n!(n-1)!k!(k-1)!} \right) \times \\ & \times \int_0^\infty e^{-(4j+1)u} u^{n-1} \int_0^\infty e^{-(4j+1)v} v^{k-1} dv dH_2(l_1, l_2). \end{aligned} \quad (4.58)$$

Since the series $S(a) := \sum_{m=1}^\infty \frac{a^m}{m!(m-1)!}$ is evidently absolutely convergent for all a , then we can change the order of summation and integration in the central integrals to obtain the following form of the above integral :

$$\begin{aligned} & \int_{\mathbb{R}_+^2} dH_2(l_1, l_2) \left(\int \int_{\mathbb{R}_+^2} \left(\frac{e^{-4(u+v)} j}{uv} \{((re(s) - 1)ul_1)\} + \sum_{n=1}^\infty \frac{[(re(s) - 0.5)ul_1]^n}{n!(n-1)!} \right) e^{-l_1/2} \times \right. \\ & \left. \left(\int_0^\infty (-im(s)vl_2) + \sum_{k=1}^\infty \frac{(-im(s)vl_2)^k}{k!(k-1)!} dv \right) = \right. \\ & \left. = \int_{\mathbb{R}_+^2} dH_2(l_1, l_2) \int \int_{\mathbb{R}_+^2} (e^{-4(u+v)})^j \frac{S((re(s) - 0.5)ul_1)S(-im(s)vl_2) du dv}{e^{l_1/2+u+v} uv} \right). \end{aligned} \quad (4.59)$$

As usual - if we play a game with RH - to obtain an interesting integral representation - we have to change an order of integration. And here is a crucial moment of the using of the **positivity** of the series $S(a)$. Then the above mentioned changing of integration is obvious - since the Fubini theorem always holds for the flat integrable positive function!. Thus, we finally get for : $1/2 < re(s) \leq 1$ and $im(s) < 0$:

$$\left| \frac{(0.5 - s)}{4j + 1} + \frac{1}{2} \right|^{-2} = \int_0^1 x^j dB_s^1(x), \quad (4.60)$$

where $B_s^1(A) := \nu_s^1(\{(u, v) \in \mathbb{R}_+^2 : e^{-4(u+v)} \in A\})$ (A is a Borel set in $[0,1]$) is the "distribution" of the "random variable" $e^{-4(u+v)}$ on the measure space $(\mathbb{R}_+^2, \nu_s^1)$, where

$$d\nu_s^1(u, v) := (uv)^{-1} \left(\int \int_{\mathbb{R}_+^2} e^{-l_1/2} S((re(s) - 0.5)ul_1) S(-im(s)vl_2) dH_2(l_1, l_2) du dv \right). \quad (4.61)$$

According to the Bernstein theorem (see (4.57)) applied in the case $l = 1$, there exists a positive measure B^2 with

$$\frac{1}{4j+1} = \int_0^1 x^j dB^2(x). \quad (4.62)$$

Finally, according to the existence of the real Laplace representation of $|z|^{-2}$, there exists a positive measure B_s^3 with the property :

$$|s+2j|^{-2} = \int_0^1 x^j dB_s^3. \quad (4.63)$$

Combining (4.60), (4.62) and (4.63) we finally get

$$t_j(s) = \int \int \int_{[0,1]^3} (xyz)^j dB_s^1(x) dB^2 dB_s^3(x) = \int_0^1 x^j dh_s(x), \quad (4.64)$$

where $h_s(A) := (B_s^1 \otimes B^2 \otimes B_s^3)(\{(x, y, z) \in [0, 1]^3 : xyz \in A\})$.

(P_4) (**The finiteness and strict positivity of zeta-traces**).

Let us introduce the following **Cauchy-Gauss traces** of zeta (see [ML]) :

$$Tr_{CG}(s) = \frac{1}{|s(s-1)|^2} + tr_{CG}(s), \quad (4.65)$$

$$tr_{CG}(s) := \sum_{n=1}^{\infty} tr_{CG}^n(s), \quad (4.66)$$

and

$$tr_{CG}^n(s) := \sum_{j=0}^{\infty} \frac{(-\pi n^2)^j}{j!} t_j(s). \quad (4.67)$$

But according to (4.64) we have

$$\begin{aligned} tr_{CG}^n(s) &= \sum_{j=0}^{\infty} \frac{(-\pi n^2)^j}{j!} \int_0^1 x^j dh_s(x) = \\ &= \int_0^1 \left(\sum_{j=0}^{\infty} \frac{(-\pi n^2 x)^j}{j!} \right) dh_s(x) = \int_0^1 e^{-\pi n^2 x} dh_s(x), \end{aligned} \quad (4.68)$$

sine the **Tonelli theorem** holds in this case. Indeed, let us introduce the sequence of functions $\{f_j\}$ by the formula : $f_j(x) := \frac{(\pi n^2 x)^j}{j!}$. Then

$$\begin{aligned} 0 &< \sum_{j=1}^{\infty} \int_0^1 |f_j(x)| dh_s(x) \leq \left(\sum_{j=0}^{\infty} \frac{(\pi n^2)^j}{j!} \right) \sup_j \left(\int_0^1 x^j dh_s(x) \right) = \\ &= e^{\pi n^2} \sup_j h_s(j) < +\infty, \end{aligned} \quad (4.69)$$

since obviously the sequence $t(s)$ is **bounded**. Thus, for each $n \geq 1$

$$tr_{CG}^n(s) = \int_0^1 e^{-\pi n^2 x} dh_s(x) > 0,$$

and in a consequence we have : $Tr_{CG}(s) > 0$.

On the other hand we have :

$$tr_{CG}^n(s) =: im(s)(tr_{CG}^{n0}(s) - tr_{CG}^{n0}(1-s)), \quad (4.70)$$

where for $z = s$ or $z = 1-s$

$$tr_{CG}^{n0}(z) := \sum_{j=0}^{\infty} \frac{(\pi n^2)^j}{j!} \cdot \frac{1}{|2j+z|^2} = \quad (4.71)$$

$$\sum_{j=0}^{\infty} \int_{\mathbb{R}_+^2} \frac{(\pi n^2)^j}{j!} (e^{-2jl_1})^j e^{-z \cdot l} dH_2(l_1, l_2) = \int_{\mathbb{R}_+^2} e^{-\pi n^2 e^{-2l_1}} e^{-z \cdot l} dH_2(l).$$

Therefore, applying the inequality $e^{-x} \leq \frac{d!}{x^d}$ for arbitrary $d \in \mathbb{N}^*, x > 0$, we obtain

$$\begin{aligned} tr_{CG}^0 &:= \sum_{n=1}^{\infty} tr_{CG}^{n0} = \\ &= \sum_{n=1}^{\infty} \int_{\mathbb{R}_+^2} e^{-\pi n^2 e^{-2l_1}} e^{-z \cdot l} dH_2(l) \leq \sum_{n=1}^{\infty} \frac{d!}{(\pi n^2)^d} \int_{\mathbb{R}_+^2} e^{-(z-2d) \cdot l} dH_2(l) = \\ &= \frac{d!}{\pi^d} \frac{\zeta(2)}{|z-2d|^2} < +\infty, \end{aligned} \quad (4.72)$$

if only $d \geq 1$ and $re(z) \in [0, 1]$.

(P_5). **The Riemann hypothesis functional equation**(Rhfe for short).

Combining all calculations from the points (P_1-P_4)- we obtain the following **Riemann hypothesis functional equation**(Rhfe in short) for $re(s) \in [1/2, 1]$ and $im(s) < 0$: let

$$\zeta^*(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

be the Gamma evoluted zeta. Then

$$im(\zeta^*(s)) = im(s)(2re(s) - 1)Tr_{CG}(s), \quad (4.73)$$

with $Tr_{CG}(s) > 0$, which obviously immediately implies the famous **Riemann hypothesis**.

Remark 4 Let us observe that $\zeta(s)$ for $\text{re}(s) > 1$ is nothing that the **complex Laplace transform** of the **zeta measure** \mathcal{Z} , i.e.

$$\zeta(s) = \int_0^\infty e^{-sx} d\mathcal{Z}(x) = \sum_{n=1}^\infty (e^{\log n})^{-s},$$

where

$$\mathcal{Z} := \sum_{n=1}^\infty \delta_{\log n},$$

and obviously δ_a is the **Dirac delta measure** at a .

In particular the **Riemann hypothesis** (*Rh* for short) is **equivalent** to the following **measure Riemann hypothesis** (*mRh*) : there is a surprising, deep and unexpected relation between the zeta measure \mathcal{Z} with $\mathcal{L}(\mathcal{Z})(s) = \zeta(s)$ and the **Lebesgue measure** d^2l with $\mathcal{L}_{\mathbb{C}}(d^2l)(z) = |z|^2$ as well as the **Hodge measure** H_2 with $\mathcal{L}_{\mathbb{R}}(H_2)(s) = |s|^{-2}$, i.e. formally,

$$\mathcal{Z} \longleftrightarrow (d^2l, dH_2).$$

That connection is expressed explicite in the following **Cauchy-Gauss-Fresnel-Riemann hypothesis equation** ($\mathcal{Z}d^2l$) :

$$\begin{aligned} & (\mathcal{Z}d^2lH_2) \quad \text{im}(\pi^{-s/2}\Gamma(\frac{s}{2}) \int_{\mathbb{R}_+} e^{-sx} d\mathcal{Z}(x)) = \\ & = \text{im}(s)(2\text{re}(s) - 1) \sum_{n=0}^\infty \sum_{j=0}^\infty \frac{(-\pi n^2)^j (4j+1)}{j!} \times \\ & \times \left(\int \int_{\mathbb{R}_+^4} e^{-(\langle 2j+s, l_1 \rangle + \langle 2j+1-s, l_2 \rangle)} d^2l_1 \otimes d^2l_2 = \int \int_{\mathbb{R}_+^4} e^{-[(2j+s) \cdot l_1 + (2j+1-s) \cdot l_2]} dH_2(l_1) \otimes dH_2(l_2) \right), \quad \text{re}(s) > 1. \end{aligned}$$

Let us also remark that the sequence $\{\frac{(4j+1)}{j!}\}$ is quite special! It appears as the sequence of coefficients in the Taylor expansion of the **canonical second Hermite function**.

Moreover, there exists also the connection

$$\mathcal{Z} \longleftrightarrow \{h_s\},$$

given by the functional equation :

$$\begin{aligned} & (\mathcal{Z}h_s) \quad \text{im}(\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)) = \text{im}(s)(2\text{re}(s) - 1) \sum_{n=0}^\infty \sum_{j=0}^\infty \frac{(-\pi n^2)^j}{j!} \int_0^1 x^j dh_s(x) = \\ & = \text{im}(s)(2\text{re}(s) - 1) \sum_{n=0}^\infty \int_0^1 e^{-\pi n^2 x} dh_s = \text{im}(s)(2\text{re}(s) - 1) \int_0^1 (\theta(x) + 1) dh_s(x) = \\ & =: \zeta_t(s) \int_0^1 (\theta(x) + 1) dh_s(x) \end{aligned}$$

, where $\zeta_t(s) := \text{im}(s)(2\text{re}(s) - 1)$ is the so called **trivial zeta**.

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